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On higher order relations in Fedosov supermanifolds

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Abstract
Higher order relations existing in normal coordinates between affine extensions of the symplectic curvature tensor and basic objects for any Fedosov supermanifolds are derived. Representation of these relations in general coordinates is discussed.

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1. Introduction

Fedosov supermanifolds are a special kind of supermanifolds introduced by Berezin [1] and studied in detail by DeWitt [2]. They are introduced as even or odd symplectic supermanifolds endowed with a symmetric connection which preserves a given symplectic structure. In the even case, they can be considered as natural extensions of Fedosov manifolds [3, 4] in the supersymmetric case. In the odd case, there is no analogue for them in differential geometry on manifolds. Note that modern quantum field theory involves symplectic supermanifolds to formulate quantization procedures. The well-known quantization method proposed by Batalin and Vilkovisky [5] is based on geometry of odd symplectic supermanifolds [6]. In turn, the deformation quantization [3] can be formulated for any even symplectic supermanifolds (see [7, 8]). Simple types of Fedosov supermanifolds have already appeared in the physical literature. Namely, flat even Fedosov supermanifolds have been used to construct a coordinate-free quantization procedure [9] and the triplectic quantization method [10, 11] in general coordinates [12].

Systematic investigation of basic properties of even and odd Fedosov supermanifolds has started in [13] and continued in [14]. In particular, the relations between a supersymplectic structure, a connection and the symplectic curvature tensor in the lowest (first and second) orders have been derived (the order of relations is defined by the order of affine extension of connection). It was shown that relations of the second order are very useful to derive new identities containing the first-order covariant derivatives of the symplectic curvature tensor. From this point of view, it seems to be very interesting to investigate higher order relations in Fedosov supermanifolds.

The goal of the present paper is to study higher order relations existing among the affine connection, symplectic structure and the symplectic curvature tensor and to find a fundamental origin of all these relations. Namely, we derive explicit forms of these relations in the third order and discuss consequences from them to obtain identities for the symplectic curvature tensor. Moreover, we state that all higher order relations can be considered as coefficients of the Taylor expansion of generating functions found in the closed form.

The paper is organized as follows. In section 2, the notion of even (odd) Fedosov supermanifolds and of even (odd) symplectic curvature tensors are given. In section 3, affine extensions of connection and tensors on a supermanifold are considered. In section 4, relations existing among affine extensions of connection, symplectic structure and the symplectic curvature tensor of the first and second orders are presented. In section 5, relations of the third order for objects listed in section 4 are derived. In section 6, generating functions for all higher order relations are found. In section 7 concluding remarks are given.

We use the condensed notation suggested by DeWitt [15]. Derivatives with respect to the coordinates x^i are understood as acting from the right and for them the notation $A_{,i} = \partial_r A/\partial x^i$ is used. Covariant derivatives are understood as acting from the right with the notation $A_{;i} = A\nabla_i$. The Grassmann parity of any quantity A is denoted by $\epsilon(A)$.

2. Fedosov supermanifolds

Consider an even (odd) symplectic supermanifold, (M, ω) with an even (odd) symplectic structure $\omega, \epsilon(\omega) = 0$ (1). Let us equip (M, ω) with a covariant derivative (or connection) ∇ (or Γ) which preserves the symplectic structure $\omega, \omega\nabla = 0$. In a coordinate basis this requirement reads

$$\omega_{ij,k} - \omega_{im}\Gamma^m{}_{jk} + \omega_{jm}\Gamma^m{}_{ik}(-1)^{\epsilon_i\epsilon_j} = 0, \qquad \omega_{ij} = -\omega_{ji}(-1)^{\epsilon_i\epsilon_j}. \tag{1}$$

If, in addition, Γ is symmetric $\Gamma^i{}_{jk} = \Gamma^i{}_{kj}(-1)^{\epsilon_k\epsilon_j}$ then the triple (M, ω, Γ) is defined as a Fedosov supermanifold.

The curvature tensor of a symplectic connection with all indices lowered

$$R_{imjk} = \omega_{in}R^n{}_{mjk} \tag{2}$$

obeys the following symmetry properties [13]:

$$R_{ijkl} = -(-1)^{\epsilon_k\epsilon_l}R_{ijlk}, \qquad R_{ijkl} = (-1)^{\epsilon_i\epsilon_j}R_{jikl}. \tag{3}$$

Using definition of tensor field ω^{ij} inverse to the symplectic structure ω_{ij} [13] and notation,

$$\Gamma_{ijk} = \omega_{in}\Gamma^n{}_{jk}, \qquad \epsilon(\Gamma_{ijk}) = \epsilon(\omega) + \epsilon_i + \epsilon_j + \epsilon_k,$$

one obtains the following representation for the symplectic curvature tensor:

$$R_{ijkl} = -\Gamma_{ijk,l} + \Gamma_{ijl,k}(-1)^{\epsilon_l\epsilon_k} + \Gamma_{nik}\Gamma^n{}_{jl}(-1)^{\epsilon_k(\epsilon_n+\epsilon_j)+\epsilon_n\epsilon_i} - \Gamma_{nil}\Gamma^n{}_{jk}(-1)^{\epsilon_l(\epsilon_n+\epsilon_j+\epsilon_k)+\epsilon_n\epsilon_i}. \tag{4}$$

The (super-)Jacobi identity for R_{ijkl} holds:

$$R_{ijkl}(-1)^{\epsilon_j\epsilon_l} + R_{iljk}(-1)^{\epsilon_l\epsilon_k} + R_{iklj}(-1)^{\epsilon_k\epsilon_j} = 0. \tag{5}$$

For any even (odd) symplectic connection there holds the identity (see [13])

$$R_{ijkl} + R_{lijk}(-1)^{\epsilon_l(\epsilon_k+\epsilon_j+\epsilon_i)} + R_{klij}(-1)^{(\epsilon_k+\epsilon_l)(\epsilon_i+\epsilon_j)} + R_{jkli}(-1)^{\epsilon_i(\epsilon_j+\epsilon_k+\epsilon_l)} = 0. \tag{6}$$

In the identity (6) the components of the symplectic curvature tensor occur with cyclic permutations of all the indices (on R). However, the pre-factors depending on the Grassmann parities of indices are not obtained by cyclic permutations as in the case of the Jacobi identity (5) but by permutations of indices from the given set to the initial one.

3. Affine extensions of tensors on supermanifolds

In [4] the virtues of using the method of normal coordinates for studying the properties of Fedosov manifolds were demonstrated. Normal coordinates $\{y^i\}$ at a point $p \in M$ can be introduced by using the geodesic equations as those local coordinates which satisfy the relations (p corresponds to $y = 0$):

$$\Gamma_{ijk}(y)y^k y^j = 0, \qquad \epsilon(\Gamma_{ijk}) = \epsilon(\omega) + \epsilon_i + \epsilon_j + \epsilon_k. \tag{7}$$

It follows from (7) and the symmetry properties of Γ_{ijk} w.r.t. (jk) that $\Gamma_{ijk}(0) = 0$. In normal coordinates there exist additional relations at p containing the partial derivatives of Γ_{ijk}. Namely, consider the Taylor expansion of $\Gamma_{ijk}(y)$ at $y = 0$,

$$\Gamma_{ijk}(y) = \sum_{n=1}^{\infty} \frac{1}{n!} A_{ijkj_1\cdots j_n} y^{j_n} \cdots y^{j_1}, \quad \text{where} \quad A_{ijkj_1\ldots j_n} = A_{ijkj_1\ldots j_n}(p) = \left.\frac{\partial_r^n \Gamma_{ijk}}{\partial y^{j_1} \ldots \partial y^{j_n}}\right|_{y=0}, \tag{8}$$

is called an affine extension of Γ_{ijk} of order $n = 1, 2, \ldots$. The symmetry properties of $A_{ijkj_1\ldots j_n}$ are evident from their definition (8); namely, they are (generalized) symmetric w.r.t. (jk) as well as $(j_1, \ldots, j_n)$. The set of all affine extensions of Γ_{ijk} uniquely defines a symmetric connection according to (8) and satisfy an infinite sequence of identities [14]. In the lowest nontrivial orders they have the form

$$A_{ijkl} + A_{ijlk}(-1)^{\epsilon_k\epsilon_l} + A_{iklj}(-1)^{\epsilon_j(\epsilon_l+\epsilon_k)} = 0, \tag{9}$$

$$\begin{aligned} A_{ijklm} &+ A_{ijlkm}(-1)^{\epsilon_k\epsilon_l} + A_{ikljm}(-1)^{\epsilon_j(\epsilon_l+\epsilon_k)} + A_{ijmkl}(-1)^{\epsilon_m(\epsilon_l+\epsilon_k)} \\ &+ A_{ilmjk}(-1)^{(\epsilon_j+\epsilon_k)(\epsilon_m+\epsilon_l)} + A_{ikmjl}(-1)^{\epsilon_j(\epsilon_m+\epsilon_k)+\epsilon_m\epsilon_l} = 0 \end{aligned} \tag{10}$$

and

$$\begin{aligned} A_{ijklmn} &+ A_{ijlkmn}(-1)^{\epsilon_k\epsilon_l} + A_{ikljmn}(-1)^{\epsilon_j(\epsilon_l+\epsilon_k)} + A_{ijmkln}(-1)^{\epsilon_m(\epsilon_l+\epsilon_k)} \\ &+ A_{ilmjkn}(-1)^{(\epsilon_j+\epsilon_k)(\epsilon_m+\epsilon_l)} + A_{ikmjln}(-1)^{\epsilon_j(\epsilon_m+\epsilon_k)+\epsilon_m\epsilon_l} \\ &+ A_{ijnklm}(-1)^{\epsilon_n(\epsilon_l+\epsilon_k+\epsilon_m)} + A_{iknjlm}(-1)^{\epsilon_j(\epsilon_n+\epsilon_k)+\epsilon_n(\epsilon_l+\epsilon_m)} \\ &+ A_{ilnjkm}(-1)^{(\epsilon_j+\epsilon_k)(\epsilon_n+\epsilon_l)+\epsilon_m\epsilon_n} + A_{imnjkl}(-1)^{(\epsilon_j+\epsilon_k+\epsilon_l)(\epsilon_m+\epsilon_n)} = 0. \end{aligned} \tag{11}$$

Analogously, the affine extensions of an arbitrary tensor $T = \left(T^{i_1\cdots i_k}{}_{m_1\cdots m_l}\right)$ on M are defined as tensors on M whose components at $p \in M$ in the local coordinates $(x^1, \ldots, x^{2N})$ are given by the formula

$$T^{i_1\cdots i_k}{}_{m_1\cdots m_l,j_1\cdots j_n} \equiv T^{i_1\cdots i_k}{}_{m_1\cdots m_l,j_1\cdots j_n}(0) = \left.\frac{\partial_r^n T^{i_1\cdots i_k}{}_{m_1\cdots m_l}}{\partial y^{j_1} \cdots \partial y^{j_n}}\right|_{y=0}$$

where $(y^1, \ldots, y^{2N})$ are normal coordinates associated with $(x^1, \ldots, x^{2N})$ at p. The first extension of any tensor coincides with its covariant derivative because $\Gamma^i{}_{jk}(0) = 0$ in normal coordinates.

In the following, any relation containing affine extensions is to be understood as holding in a neighbourhood U of an arbitrary point $p \in M$. Let us also observe the convention that, if a relation holds for arbitrary local coordinates, the arguments of the related quantities will be suppressed. The order of relations is defined by the order of affine extension of Γ_{ijk} entering in the relations.

4. First- and second-order relations

For a given Fedosov supermanifold (M, ω, Γ), symmetric connection Γ respects the symplectic structure ω:

$$\omega_{ij,k} = \Gamma_{ijk} - \Gamma_{jik}(-1)^{\epsilon_i\epsilon_j}. \tag{12}$$

Taking into account (12) and comparing (8) and the Taylor expansion for $\omega_{ij,k}(y)$ we obtain

$$\omega_{ij,kj_1\ldots j_n}(0) = A_{ijkj_1\ldots j_n} - A_{jikj_1\ldots j_n}(-1)^{\epsilon_i\epsilon_j}. \tag{13}$$

Now, consider the symplectic curvature tensor R_{ijkl} in the normal coordinates at $p \in M$. Then, due to $\Gamma_{ijk}(p) = 0$, we obtain the following representation of the symplectic curvature tensor in terms of the affine extensions of Γ_{ijk}:

$$R_{ijkl}(0) = -A_{ijkl} + A_{ijlk}(-1)^{\epsilon_k\epsilon_l}. \tag{14}$$

From (9) and (14) a relation containing the symplectic curvature tensor and the first affine extension of Γ can be derived. Indeed, the desired relation is obtained as follows,

$$A_{ijkl} \equiv \Gamma_{ijk,l}(0) = -\tfrac{1}{3}\left[R_{ijkl}(0) + R_{ikjl}(0)(-1)^{\epsilon_k\epsilon_j}\right], \tag{15}$$

where the antisymmetry (3) of the symplectic curvature tensor was used.

Taking into account (13) and (14) the relation between the second-order affine extension of symplectic structure and the symplectic curvature tensor follows. Indeed, using the Jacobi identity (5), we obtain

$$\omega_{ij,kl}(0) = A_{ijkl} - A_{jikl}(-1)^{\epsilon_i\epsilon_j} = \tfrac{1}{3}R_{klij}(0)(-1)^{(\epsilon_i+\epsilon_j)(\epsilon_k+\epsilon_l)}. \tag{16}$$

Symmetry properties of $R_{klij}(0)$ and $\omega_{ij,kl}(0)$ are in accordance with this relation.

The second-order affine extension of Γ_{ijk} in terms of first-order derivatives of the symplectic curvature tensor is given by the formula

$$\begin{aligned} A_{ijklm} = -\tfrac{1}{6}[&2R_{ijkl,m}(0) + R_{ijkm,l}(0)(-1)^{\epsilon_m\epsilon_l} + R_{ikjl,m}(0)(-1)^{\epsilon_j\epsilon_k} \\ &+ R_{ikjm,l}(0)(-1)^{\epsilon_j\epsilon_k+\epsilon_m\epsilon_l} + R_{iljm,k}(0)(-1)^{\epsilon_j\epsilon_l+\epsilon_k(\epsilon_m+\epsilon_l)}]. \end{aligned} \tag{17}$$

Using (13) we get

$$\begin{aligned} \omega_{ij,klm}(0) = -\tfrac{1}{6}[&R_{ikjl,m}(0)(-1)^{\epsilon_j\epsilon_k} + R_{ikjm,l}(0)(-1)^{\epsilon_j\epsilon_k+\epsilon_m\epsilon_l} + R_{iljm,k}(0)(-1)^{\epsilon_j\epsilon_l+\epsilon_k(\epsilon_l+\epsilon_m)} \\ &- R_{jkil,m}(0)(-1)^{\epsilon_i(\epsilon_k+\epsilon_j)} - R_{jkim,l}(0)(-1)^{\epsilon_m\epsilon_l+\epsilon_i(\epsilon_j+\epsilon_k)} \\ &- R_{jlim,k}(0)(-1)^{\epsilon_k(\epsilon_m+\epsilon_l)+\epsilon_i(\epsilon_j+\epsilon_l)}] \end{aligned} \tag{18}$$

as the representation of the third-order affine extensions of ω_{ij} in terms of the first-order affine extension of the symplectic curvature tensor. Identities for the first affine extension of the symplectic curvature tensor have the form

$$\begin{aligned} &R_{mjik,l}(0)(-1)^{\epsilon_j(\epsilon_i+\epsilon_k)} - R_{mijl,k}(0)(-1)^{\epsilon_k(\epsilon_l+\epsilon_j)} + R_{mkjl,i}(0)(-1)^{\epsilon_i(\epsilon_j+\epsilon_k+\epsilon_l)} \\ &\quad - R_{mlik,j}(0)(-1)^{\epsilon_l(\epsilon_i+\epsilon_j+\epsilon_k)} = 0, \end{aligned} \tag{19}$$

and are derived from the representation (18) and the symmetry of $\omega_{ik,jlm}$ [14].

5. Third-order relations

Beginning with the third-order relations, we meet a new feature concerning representation of affine extensions of the symplectic curvature tensor and the symplectic structure in terms of affine extensions of connection. This feature is connected with nonlinear dependence in contrast with relations of the first and second orders. Indeed, from (4) the following representation for the second-order extension of the symplectic curvature tensor can be derived,

$$R_{ijkl,mn}(0) = -A_{ijklmn} + A_{ijlkmn}(-1)^{\epsilon_k\epsilon_l} + N_{ijklmn}, \tag{20}$$

where

$$N_{ijklmn} = T_{ijklmn} + T_{ijklnm}(-1)^{\epsilon_n\epsilon_m} - T_{ijlkmn}(-1)^{\epsilon_k\epsilon_l} - T_{ijlknm}(-1)^{\epsilon_k\epsilon_l+\epsilon_n\epsilon_m}, \tag{21}$$

$$T_{ijklmn} = A_{sikm}A^s{}_{jln}(-1)^{\epsilon_s\epsilon_i+\epsilon_k(\epsilon_s+\epsilon_j)+\epsilon_m(\epsilon_l+\epsilon_j+\epsilon_s)} \tag{22}$$

is quadratic in the first-order extension of connection. In (21) we used the notation

$$A^i{}_{jkl} = \omega^{ip}A_{pjkl}(-1)^{\epsilon_p+\epsilon(\omega)(\epsilon_p+\epsilon_i)} = -\tfrac{1}{3}\big[R^i{}_{jkl}(0) + R^i{}_{kjl}(0)(-1)^{\epsilon_k\epsilon_j}\big]. \tag{23}$$

From (22) the symmetry properties follow:

$$T_{ijklmn} = T_{ilkjmn}(-1)^{\epsilon_l(\epsilon_k+\epsilon_j)+\epsilon_j\epsilon_k}, \tag{24}$$

$$T_{ijklmn} = T_{kjilmn}(-1)^{\epsilon_k(\epsilon_i+\epsilon_j)+\epsilon_i\epsilon_j}, \tag{25}$$

$$T_{ijklmn} = -T_{jilknm}(-1)^{\epsilon_i\epsilon_j+\epsilon_k\epsilon_l+\epsilon_m\epsilon_n}. \tag{26}$$

Taking into account the symmetry properties of A_{ijklmn} and (20) we have

$$\begin{aligned}
A_{ijklmn}(-1)^{\epsilon_k\epsilon_l} &= R_{ijkl,mn}(0) + A_{ijklmn} - N_{ijklmn},\\
A_{ijmkln}(-1)^{\epsilon_k\epsilon_m} &= R_{ijkm,ln}(0) + A_{ijkmln} - N_{ijkmln} = R_{ijkm,ln}(0) + A_{ijklmn}(-1)^{\epsilon_m\epsilon_l} - N_{ijkmln},\\
A_{ijnklm}(-1)^{\epsilon_k\epsilon_n} &= R_{ijkn,lm}(0) + A_{ijknlm} - N_{ijknlm}\\
&= R_{ijkn,lm}(0) + A_{ijklmn}(-1)^{\epsilon_l\epsilon_n+\epsilon_m\epsilon_n} - N_{ijknlm},\\
A_{ikljmn}(-1)^{\epsilon_l\epsilon_j} &= R_{ikjl,mn}(0) + A_{ikjlmn} - N_{ikjlmn} = R_{ikjl,mn}(0) + A_{ijklmn}(-1)^{\epsilon_j\epsilon_k} - N_{ikjlmn},\\
A_{ikmjln}(-1)^{\epsilon_m\epsilon_j} &= R_{ikjm,ln}(0) + A_{ikjmln} - N_{ikjmln}\\
&= R_{ikjm,ln}(0) + A_{ijklmn}(-1)^{\epsilon_k\epsilon_j+\epsilon_m\epsilon_l} - N_{ikjmln},\\
A_{iknjlm}(-1)^{\epsilon_j\epsilon_n} &= R_{ikjn,lm}(0) + A_{ikjnlm} - N_{ikjnlm}\\
&= R_{ikjn,lm}(0) + A_{ijklmn}(-1)^{\epsilon_j\epsilon_k+\epsilon_n(\epsilon_m+\epsilon_l)} - N_{ikjnlm},\\
A_{ilmjkn}(-1)^{\epsilon_j\epsilon_m} &= R_{iljm,kn}(0) + A_{iljmkn} - N_{iljmkn}\\
&= R_{iljm,kn}(0) + A_{ijlkmn}(-1)^{\epsilon_j\epsilon_l+\epsilon_k\epsilon_m} - N_{iljmkn}\\
&= R_{iljm,kn}(0) + (R_{ijkl,mn} + A_{ijklmn} - N_{ijklmn})(-1)^{\epsilon_j\epsilon_l+\epsilon_k\epsilon_l+\epsilon_k\epsilon_m} - N_{iljmkn},\\
A_{ilnjkm}(-1)^{\epsilon_j\epsilon_n} &= R_{iljn,km}(0) + A_{iljnkm} - N_{iljnkm}\\
&= R_{iljn,km}(0) + A_{ijlkmn}(-1)^{\epsilon_j\epsilon_l+\epsilon_k\epsilon_n+\epsilon_m\epsilon_n} - N_{iljnkm}\\
&= R_{iljn,km}(0) + (R_{ijkl,mn} + A_{ijklmn} - N_{ijklmn})(-1)^{\epsilon_j\epsilon_l+\epsilon_k\epsilon_l+\epsilon_k\epsilon_n+\epsilon_m\epsilon_n} - N_{iljnkm},\\
A_{imnjkl}(-1)^{\epsilon_j\epsilon_n} &= R_{imjn,kl}(0) + A_{imjnkl} - N_{imjnkl}\\
&= R_{imjn,kl}(0) + A_{ijmknl}(-1)^{\epsilon_j\epsilon_m+\epsilon_k\epsilon_n} - N_{imjnkl}\\
&= R_{iljn,km}(0) + (R_{ijkl,mn} + A_{ijklmn} - N_{ijklmn})(-1)^{\epsilon_j\epsilon_m+\epsilon_k\epsilon_m+\epsilon_k\epsilon_n} - N_{imjnkl}.
\end{aligned}$$

Putting them into the identity (11) we obtain

$$\begin{aligned}
10A_{ijklmn} &+ 3R_{ijkl,mn} + 2R_{ijkm,ln}(-1)^{\epsilon_m\epsilon_l} + R_{ikjl,mn}(-1)^{\epsilon_j\epsilon_k} + R_{ijkn,lm}(-1)^{\epsilon_n(\epsilon_m+\epsilon_l)}\\
&+ R_{ikjn,lm}(-1)^{\epsilon_j\epsilon_k+\epsilon_n(\epsilon_m+\epsilon_l)} + R_{ikjm,ln}(-1)^{\epsilon_j\epsilon_k+\epsilon_m\epsilon_l} + R_{iljm,kn}(-1)^{\epsilon_j\epsilon_l+\epsilon_k\epsilon_l+\epsilon_m\epsilon_k}\\
&+ R_{iljn,km}(-1)^{\epsilon_l(\epsilon_k+\epsilon_j)+\epsilon_n(\epsilon_k+\epsilon_m)} + R_{imjn,kl}(-1)^{\epsilon_j\epsilon_m+(\epsilon_l+\epsilon_k)(\epsilon_n+\epsilon_m)} - 3N_{ijklmn}\\
&- N_{ijkmln}(-1)^{\epsilon_m\epsilon_l} - N_{ijknlm}(-1)^{\epsilon_n(\epsilon_m+\epsilon_l)} - N_{ikjlmn}(-1)^{\epsilon_j\epsilon_k}\\
&- N_{ikjmln}(-1)^{\epsilon_j\epsilon_k+\epsilon_m\epsilon_l} - N_{ikjnlm}(-1)^{\epsilon_j\epsilon_k+\epsilon_n(\epsilon_m+\epsilon_l)} - N_{iljmkn}(-1)^{\epsilon_j\epsilon_l+\epsilon_k\epsilon_l+\epsilon_m\epsilon_k}\\
&- N_{iljnkm}(-1)^{\epsilon_l(\epsilon_k+\epsilon_j)+\epsilon_n(\epsilon_k+\epsilon_m)} - N_{ijkmnl}(-1)^{\epsilon_l(\epsilon_m+\epsilon_n)}\\
&- N_{imjnkl}(-1)^{\epsilon_j\epsilon_m+(\epsilon_l+\epsilon_k)(\epsilon_n+\epsilon_m)} = 0.
\end{aligned}$$

Therefore, we have the nonlinear representation of the third-order extension of affine connection in terms of the symplectic curvature tensor:

$$
\begin{aligned}
A_{ijklmn} = -\tfrac{1}{10}[&3R_{ijkl,mn} + 2R_{ijkm,ln}(-1)^{\epsilon_m\epsilon_l} + R_{ikjl,mn}(-1)^{\epsilon_j\epsilon_k} + R_{ijkn,lm}(-1)^{\epsilon_n(\epsilon_m+\epsilon_l)} \\
&+ R_{ikjn,lm}(-1)^{\epsilon_j\epsilon_k+\epsilon_n(\epsilon_m+\epsilon_l)} + R_{ikjm,ln}(-1)^{\epsilon_j\epsilon_k+\epsilon_m\epsilon_l} + R_{iljm,kn}(-1)^{\epsilon_j\epsilon_l+\epsilon_k\epsilon_l+\epsilon_m\epsilon_k} \\
&+ R_{iljn,km}(-1)^{\epsilon_l(\epsilon_k+\epsilon_j)+\epsilon_n(\epsilon_k+\epsilon_m)} + R_{imjn,kl}(-1)^{\epsilon_j\epsilon_m+(\epsilon_l+\epsilon_k)(\epsilon_n+\epsilon_m)} \\
&- 3N_{ijklmn} - N_{ijkmln}(-1)^{\epsilon_m\epsilon_l} - N_{ijknlm}(-1)^{\epsilon_n(\epsilon_m+\epsilon_l)} - N_{ikjlmn}(-1)^{\epsilon_j\epsilon_k} \\
&- N_{ikjmln}(-1)^{\epsilon_j\epsilon_k+\epsilon_m\epsilon_l} - N_{ikjnlm}(-1)^{\epsilon_j\epsilon_k+\epsilon_n(\epsilon_m+\epsilon_l)} - N_{iljmkn}(-1)^{\epsilon_j\epsilon_l+\epsilon_k\epsilon_l+\epsilon_m\epsilon_k} \\
&- N_{iljnkm}(-1)^{\epsilon_l(\epsilon_k+\epsilon_j)+\epsilon_n(\epsilon_k+\epsilon_m)} - N_{ijkmnl}(-1)^{\epsilon_l(\epsilon_m+\epsilon_n)} \\
&- N_{imjnkl}(-1)^{\epsilon_j\epsilon_m+(\epsilon_l+\epsilon_k)(\epsilon_n+\epsilon_m)}].
\end{aligned}
\tag{27}
$$

Taking into account the relation between affine extensions of the symplectic structure and connection,

$$
\omega_{ij,klmn} = A_{ijklmn} - A_{jiklmn}(-1)^{\epsilon_i\epsilon_j},
$$

we derive the following formula for the fourth-order affine extension of the symplectic structure in terms of the symplectic curvature tensor:

$$
\begin{aligned}
\omega_{ij,klmn} = -\tfrac{1}{10}[&R_{ikjl,mn}(-1)^{\epsilon_j\epsilon_k} - R_{jkil,mn}(-1)^{\epsilon_i(\epsilon_k+\epsilon_j)} + R_{imjn,kl}(-1)^{\epsilon_j\epsilon_m+(\epsilon_l+\epsilon_k)(\epsilon_n+\epsilon_m)} \\
&- R_{jmin,kl}(-1)^{\epsilon_i(\epsilon_m+\epsilon_j)+(\epsilon_l+\epsilon_k)(\epsilon_n+\epsilon_m)} + R_{ikjn,lm}(-1)^{\epsilon_j\epsilon_k+\epsilon_n(\epsilon_m+\epsilon_l)} \\
&- R_{jkin,lm}(-1)^{\epsilon_i(\epsilon_k+\epsilon_j)+\epsilon_n(\epsilon_m+\epsilon_l)} + R_{ikjm,ln}(-1)^{\epsilon_j\epsilon_k+\epsilon_m\epsilon_l} \\
&- R_{jkim,ln}(-1)^{\epsilon_i(\epsilon_k+\epsilon_j)+\epsilon_m\epsilon_l} + R_{iljm,kn}(-1)^{\epsilon_j\epsilon_l+\epsilon_k(\epsilon_l+\epsilon_m)} \\
&- R_{jlim,kn}(-1)^{\epsilon_i(\epsilon_l+\epsilon_j)+\epsilon_k(\epsilon_l+\epsilon_m)} + R_{iljn,km}(-1)^{\epsilon_l(\epsilon_k+\epsilon_j)+\epsilon_n(\epsilon_k+\epsilon_m)} \\
&- R_{jlin,km}(-1)^{\epsilon_i\epsilon_j+\epsilon_l(\epsilon_k+\epsilon_i)+\epsilon_n(\epsilon_k+\epsilon_m)} + R_{imjn,kl}(-1)^{\epsilon_j\epsilon_m+(\epsilon_l+\epsilon_k)(\epsilon_n+\epsilon_m)} \\
&- R_{jmin,kl}(-1)^{\epsilon_i(\epsilon_m+\epsilon_j)+(\epsilon_l+\epsilon_k)(\epsilon_n+\epsilon_m)} - N_{ikjlmn}(-1)^{\epsilon_j\epsilon_k} + N_{jkilmn}(-1)^{\epsilon_i(\epsilon_k+\epsilon_j)} \\
&- N_{ikjmln}(-1)^{\epsilon_j\epsilon_k+\epsilon_m\epsilon_l} + N_{jkimln}(-1)^{\epsilon_i(\epsilon_k+\epsilon_j)+\epsilon_m\epsilon_l} - N_{ikjnlm}(-1)^{\epsilon_j\epsilon_k+\epsilon_n(\epsilon_m+\epsilon_l)} \\
&+ N_{jkinlm}(-1)^{\epsilon_i(\epsilon_k+\epsilon_j)+\epsilon_n(\epsilon_m+\epsilon_l)} - N_{iljmkn}(-1)^{\epsilon_j\epsilon_l+\epsilon_k(\epsilon_l+\epsilon_m)} \\
&+ N_{jlimkn}(-1)^{\epsilon_i(\epsilon_l+\epsilon_j)+\epsilon_k(\epsilon_l+\epsilon_m)} - N_{iljnkm}(-1)^{\epsilon_l(\epsilon_k+\epsilon_j)+\epsilon_n(\epsilon_k+\epsilon_m)} \\
&+ N_{jlinkm}(-1)^{\epsilon_i\epsilon_j+\epsilon_l(\epsilon_k+\epsilon_i)+\epsilon_n(\epsilon_k+\epsilon_m)} - N_{imjnkl}(-1)^{\epsilon_j\epsilon_m+(\epsilon_l+\epsilon_k)(\epsilon_n+\epsilon_m)} \\
&+ N_{jminkl}(-1)^{\epsilon_i(\epsilon_j+\epsilon_m)+(\epsilon_l+\epsilon_k)(\epsilon_n+\epsilon_m)}].
\end{aligned}
\tag{28}
$$

We have already found that symmetry properties of the third-order affine extension of symplectic structure expressed in terms of the symplectic curvature tensor (18) led to the new identity (19) for the curvature tensor. A natural question appears: are there some new identities containing the second-order affine extension of the symplectic curvature tensor as consequences of the representation (28) and symmetry properties,

$$
\omega_{ij,klmn} - \omega_{ij,lkmn}(-1)^{\epsilon_k\epsilon_l} = 0,
$$

of the fourth-order affine extension of the symplectic structure? We shall prove that the answer is negative. Indeed, using the symmetry properties of R_{ijkl} (2) and T_{ijklmn} (24)–(26) we have

$$
\begin{aligned}
0 = \omega_{ij,klmn} - \omega_{ij,lkmn}(-1)^{\epsilon_k\epsilon_l} &= [R_{ikjl,mn} + R_{likj,mn}(-1)^{\epsilon_l(\epsilon_i+\epsilon_j+\epsilon_k)} \\
&\quad + R_{kjli,mn}(-1)^{\epsilon_i(\epsilon_j+\epsilon_k+\epsilon_l)} + R_{jlik,mn}(-1)^{(\epsilon_i+\epsilon_k)(\epsilon_j+\epsilon_l)}](-1)^{\epsilon_j\epsilon_k} \\
&= [R_{ikjl} + R_{likj}(-1)^{\epsilon_l(\epsilon_i+\epsilon_j+\epsilon_k)} + R_{kjli}(-1)^{\epsilon_i(\epsilon_j+\epsilon_k+\epsilon_l)} \\
&\quad + R_{jlik}(-1)^{(\epsilon_i+\epsilon_k)(\epsilon_j+\epsilon_l)}]_{,mn}(-1)^{\epsilon_j\epsilon_k}.
\end{aligned}
\tag{29}
$$

Due to the identity (6) the relations (29) are satisfied identically and there are no new identities containing the second-order affine extension of the symplectic curvature tensor.

In a similar way, it is possible to find relations containing higher order affine extensions of symplectic structure, connection and the symplectic curvature tensor.

6. Higher order relations in general coordinates

Note that relations (15), (16), (18), (17), (27), (28) were derived in normal coordinates. It seems to be of general interest to find its analogue in terms of arbitrary local coordinates (x) because a connection is not a tensor while the rhs of (15) is the tensor. It means that the lhs of (15) should be a tensor G_{ijkl} taking the form $\Gamma_{ijk,l}(0)$ at point $p \in M$ in normal coordinates. In normal coordinates, the covariant derivative has the form of the usual partial one, but simple identification of G_{ijkl} with covariant derivative $\Gamma_{ijk;l}$ is wrong because it does not transform according to tensor rules. Indeed, under the change of coordinates $(x) \to (y)$ Γ_{ijk} are transformed according to the rule:

$$\Gamma_{ijk}(y) = \left(\Gamma_{pqr}(x) \frac{\partial_r x^r}{\partial y^k} \frac{\partial_r x^q}{\partial y^j} (-1)^{\epsilon_k(\epsilon_j+\epsilon_q)} + \omega_{pq}(x) \frac{\partial_r^2 x^q}{\partial y^j \partial y^k} \right) \frac{\partial_r x^p}{\partial y^i} (-1)^{(\epsilon_k+\epsilon_j)(\epsilon_i+\epsilon_p)}. \tag{30}$$

Using (30) it is easy to find the relation between derivatives of affine connection at point p in normal coordinates and arbitrary ones:

$$\Gamma_{ijk,l}(0) = \Gamma_{ijk,l}(x_0) - \tfrac{1}{3} X_{ijkl}(x_0). \tag{31}$$

Therefore, we have the representation of tensor G_{ijkl} in terms of partial derivatives of the affine connection

$$G_{ijkl}(x) = \left[\Gamma_{ijk,l} - \tfrac{1}{3} X_{ijkl} \right](x). \tag{32}$$

In (31), (32) the notation

$$\begin{aligned} X_{ijkl} = \Gamma_{ijk,l} &+ \Gamma_{ijl,k}(-1)^{\epsilon_k\epsilon_l} + \Gamma_{ikl,j}(-1)^{(\epsilon_k+\epsilon_l)\epsilon_j} + 2\Gamma_{njk}\Gamma^n{}_{il}(-1)^{(\epsilon_k+\epsilon_j)(\epsilon_i+\epsilon_n)} \\ &- \Gamma_{njl}\Gamma^n{}_{ik}(-1)^{(\epsilon_j+\epsilon_l)(\epsilon_i+\epsilon_n)+\epsilon_k\epsilon_l} - \Gamma_{nkl}\Gamma^n{}_{ij}(-1)^{(\epsilon_k+\epsilon_l)(\epsilon_i+\epsilon_n+\epsilon_j)} \end{aligned} \tag{33}$$

is used.

Now we can find some relations of the first order (15), (16) in arbitrary coordinates:

$$\Gamma_{ijk,l} - \tfrac{1}{3} X_{ijkl} = -\tfrac{1}{3}[R_{ijkl} + R_{ikjl}(-1)^{\epsilon_k\epsilon_j}], \tag{34}$$

$$\omega_{ij,kl} - \tfrac{1}{3}(X_{ijkl} - X_{jikl}(-1)^{\epsilon_i\epsilon_j}) = \tfrac{1}{3} R_{klij}(-1)^{(\epsilon_i+\epsilon_j)(\epsilon_k+\epsilon_l)}. \tag{35}$$

The difference of X in (35) is symmetric w.r.t. two last indices because of the property

$$\begin{aligned} X_{ijlk} - X_{ijkl}(-1)^{\epsilon_k\epsilon_l} &= 3\left[\Gamma_{njl}\Gamma^n{}_{ik}(-1)^{(\epsilon_j+\epsilon_l)(\epsilon_i+\epsilon_n)} - \Gamma_{njk}\Gamma^n{}_{il}(-1)^{(\epsilon_k+\epsilon_j)(\epsilon_i+\epsilon_n)+\epsilon_k\epsilon_l}\right] \\ &= -3\left[\Gamma_{nik}\Gamma^n{}_{jl}(-1)^{\epsilon_n(\epsilon_i+\epsilon_k)+\epsilon_k(\epsilon_j+\epsilon_l)} + \Gamma_{njk}\Gamma^n{}_{il}(-1)^{\epsilon_n(\epsilon_k+\epsilon_j)+\epsilon_k(\epsilon_i+\epsilon_l)+\epsilon_i\epsilon_j}\right]. \end{aligned} \tag{36}$$

It is obvious that in general coordinates the identity (19) has the form

$$R_{mjik;l}(-1)^{\epsilon_j(\epsilon_i+\epsilon_k)} - R_{mijl;k}(-1)^{\epsilon_k(\epsilon_l+\epsilon_j)} + R_{mkjl;i}(-1)^{\epsilon_i(\epsilon_j+\epsilon_k+\epsilon_l)} - R_{mlik;j}(-1)^{\epsilon_l(\epsilon_i+\epsilon_j+\epsilon_k)} = 0. \tag{37}$$

It is not difficult to check that relation (18) can be derived from (35) written in normal coordinates by differentiation w.r.t. y and then putting $y = 0$. Moreover, all higher order relations can be obtained as coefficients of the Taylor expansion of generating functions $3F_{ijkl} = 3\omega_{ij,kl} - (X_{ijkl} - X_{jikl}(-1)^{\epsilon_i\epsilon_j}) - R_{klij}(-1)^{(\epsilon_i+\epsilon_j)(\epsilon_k+\epsilon_l)} = 0$ which are considered in normal coordinates.

7. Discussion

We have considered properties of Fedosov supermanifolds. Using normal coordinates on a supermanifold, we have found relations up to the third order among the affine extensions of connection and the symplectic curvature tensor, the affine extensions of symplectic structure and the symplectic curvature tensor as well as identities for the symplectic curvature tensor. Considering relations of the third order, it was checked that there are no new independent identities containing the second-order (covariant) derivatives of the symplectic curvature tensor. In fact the identities (5), (6), (37) together with the (super-)Bianchi identity, $R_{nmjk;i}(-1)^{\epsilon_i\epsilon_j} + R_{nmij;k}(-1)^{\epsilon_i\epsilon_k} + R_{nmki;j}(-1)^{\epsilon_i\epsilon_k} \equiv 0$, describe all independent ones existing in symplectic geometry for the symplectic curvature tensor. It was shown that the tensor G_{ijkl} is very important to obtain the relations in general local coordinates. In fact, equations (35) should be considered as the fundamental ones to derive all higher order relations. Note that the tensor of such kind is not specific for symplectic geometry only and can be introduced in both affine and Riemannian geometries.

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